Logic for Serious Database Folks Series

by David McGoveran, Alternative Technologies

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Chapter 2: Set Theory and Meta-Language
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"*The limits of my language mean the limits of my world.*" -- Ludwig Wittgenstein

I. INTRODUCTION

As discussed in the previous article, the language that one uses to talk about or analyze a formal system is called a *meta-language*. Inasmuch as a significant portion of this series will involve talking about formal systems, we clearly need a meta-language. Set theory is extremely important to mathematics and logic, and in the discussion that follows a very restricted portion of the language of set theory will be used informally to explain the structure of formal systems.

The languages and formalisms of set theory can be used to define almost all other mathematical objects[1] and the symbols of logic range over the members of sets. It is perhaps fair to say that the language of set theory is the canonical meta-language for formal systems of logic and mathematics. *We will use American English augmented with certain portions of the language of set theory as our informal meta-language throughout this series.* Although we frequently speak as if there were exactly one formal system called set theory, in fact there are many variations. Most of this section addresses common concepts that apply to most of the variations. When a specific variation is relevant, it will be identified by name.

From time to time we will make reference to certain epistemological and ontological issues. Just to be clear, an epistemological issue is one that addresses matters of knowledge such as how we know and what we can know. By contrast, an ontological issue is on that addresses matters of existence including what we assume about existence. When we make an assertion on which subsequent philosophical arguments rely, we are said to make a commitment – in particular,

[1] In other words, having defined set theory appropriately, we can go on to define the rest of mathematics. Of course, the language of set theory includes the definitions of sets. There is no risk of circular definition.

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either an *epistemological commitment* or an *ontological commitment*. Such commitments are fundamentally unprovable. Whereas two or more assumptions might be found to be contradictory, commitments are independent of each other and of the specifics of the formal logical system within which one reasons. They shape subsequent philosophical arguments and how one interprets a formal system. By contrast, a mere assumption is – at least in principle – disprovable.

As we will see in the next article, every formal logical system comprises a symbolic language, a set of axioms, a set of inference rules, and certain other components. Every component is either *syntactic* or *semantic*, a crucial distinction on which we will begin to make in this article and continue to elaborate in future articles.

II. A BRIEF INTRODUCTION TO THE CONCEPTS OF SET THEORY

Any theory that pertains to the study of the abstract concept called a *set* (defined below) is known as a set theory. There are many kinds of set theory, some of which – the so-called *naïve set theories* – are explained in informal language, while using symbols only informally. *Formal set theories* define terms using a formal language with which to study sets.

If the formal language regarding sets is symbolic and augmented with *axioms*[2] and *rules of inference*[3] (both of which we will return to later), we end up with a formal logical system in general, and an *axiomatic set theory* in particular. Anticipating the relationship of set theory to logic, the axioms of a simplistic set theory such as we will be assuming may be mapped (with suitable change of terminology) one-to-one to those for a formal system of logic (namely, propositional logic). Similar variations apply regarding the rules of inference. More sophisticated set theories are similar to some variation of predicate logic.

Depending on the set theory, sets can, for example, have a finite or an infinite number of members, can be continuous or discrete, can contain *objects*[4] that are more primitive than sets (i.e., *individuals*), can contain other sets or not, and can contain themselves or not[5]. The ***cardinality*** of a set is a measure of a set's size, which may be a specific number if the set is both finite and discrete, or infinity (in which case it may be either discrete or continuous).[6] Depending

[2] For now, you may understand "axiom" as a formal, unquestioned assumption. Until interpreted, it has no epistemological weight. Interpreted, it represents an epistemological commitment within the system.

[3] For now, you may understand "rule of inference" as a formal justification for drawing a conclusion.

[4] Throughout, the term ***object*** should be understood as denoting an arbitrary structure, however formally or informally defined and however abstract or concrete. Put more colloquially, an object is a denotable "thing".

[5] As we will see when we discuss paradoxes, this is one formal system property that often leads to self-referential expressions.

[6] Many set theories also identify a relative measure of set size called *ordinality* and introduce orders of infinity, but we will not have need of such theories.

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on the power of the particular set theory or lack thereof, the concept of cardinality may or may not have been formally defined within the theory. It is tempting to define cardinality as a count of the set's members, but two objections to this come immediately to mind. First, counting is itself an abstraction – we count objects of some kind such as apples or oranges while ignoring other kinds of objects (such as rocks) and we come to understand the cardinal numbers through such abstraction. Second, if the set is continuous the abstraction is even more difficult to explain and we rely on the set being bounded and contained within some bounded context even when we say the set's cardinality is the non-numeric concept we call *infinite*.

Simple Set Theory

The present discussion will be confined to what we will here call ***simple set theory***. We define a simple set theory as kind of an informal, naïve, elementary set theory with very limited *expressive power*. The language of simple set theory consists of symbols for sets, members of sets, the binary relationship of set membership, and certain operators (union, intersection, and complement) that act on sets and produce other sets. The members of sets may or may not themselves be sets. Sets are assumed to be *well-defined* so that its members are never defined in terms of the set. When members are not sets, we refer to them as ***individuals***. Individuals are conceptually primitive and never defined in terms of sets or other individuals.

For the present purposes, you can think of simple set theory, for now, as the set theory you may have learned in school using Venn diagrams. Unless we state otherwise, the sets we consider will be both discrete and finite – the cardinality (number of members) of any set will be countable and correspond to some natural number. The cardinality of a set will not be the result of any operator defined within our set theory, but rather it will be a *primitive property* of a set[7]. A ***primitive property*** is one that, by convention, agreement, or caveat, is not analyzable into component properties, and hence is to be contrasted with *derived properties* (defined below). We won't need to define set theory in formal terms. Instead of studying set theory in its own right, we will be using set theory as part of an informal meta-language with which to present concepts about formal systems. Additionally, we will be borrowing some of the language of set theory to explain the *object languages* of some formal system exemplars, especially propositional logic.

Defining Sets

A ***set*** is an identification of zero or more *objects* (depending on context, the terms "elements" or even "entities"[8] may be used) that can be referred to as a group by name (usually a symbol) and which are drawn from some pre-defined ***universe*** of objects. Such objects are then said to be the ***members*** of the set. Usually that universe is itself considered to be, *a priori*, a kind of

[7] In many axiomatic set theories, cardinality is a defined concept resulting from a defined procedure or operation.
[8] The word "entity" is used in the literature in many ways. Herein, an ***entity*** is something that can be identified and which belongs to the subject language.

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penultimate set. In fact, it encompasses or contains all the members of all the sets one will be discussing. Sometimes a single notation (e.g., {3,9,17, -4, 0}) is used both to specify a set and as a name for the set, a practice which is denigrated and often leads to confusion.

As noted above, an individually identified object belonging to a set of objects is called a ***member*** of the set and collectively those objects are the set's *members*. ***Set membership*** is a binary relationship between an object and a set. Each of a set's members must be distinguishable from every other member of the set, and no such member may appear more than once. This requirement is sometimes and somewhat erroneously understood to mean that the members of a set are not ordered. Properly, we say that the members of a set may not be distinguished from each other merely by their order of appearance in any representation of the set: thus, the set {0, 1, 3, 1, 2} is identical to the set {0, 1, 3, 2} and to the set {0, 1, 2, 3}. When *representing* a set, we identify each member of the set with a unique symbol. Note that the inability to use order in the representation of a set as a distinguishing property does not mean that there cannot be an ordering among the members of a set. To the contrary, such sets are common in mathematics and are called ***ordered sets***. Ordered sets are sets to which an ordering has been applied.[9] The sets of natural numbers, integers, rational numbers, algebraic numbers, real numbers, and so on are all examples of ordered sets. However, such orderings may not be part of the *definition* of the set.

The members of a set may themselves be either sets or non-set *individuals*. The members of a set have one or more properties in common, if nothing else – and rather abstractly – the fact that they have each been asserted to be members of the set.[10] To make the point more explicit, for any set, an assertion of set membership has definitional priority over the necessity of any other properties being shared among the members. Much of the discussion in this section will be about how to make the definition of a set *effective*, meaning the set's members are specifiable through a finite number of steps.

Set vs. Class

Though the terms set, collection, and class are often used interchangeably, we will not engage in this practice. We will reserve collection and class for somewhat different uses. In particular, by a ***collection*** we will mean one or more arbitrary objects (any of which may be sets or individuals) that are referred to jointly, either by name or description. We will reserve the term ***class*** for a collection of objects (possibly sets) which have some definable (in whatever language is appropriate to the discussion at hand) property – possibly compound – in common.[11] A proper

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[9] Mathematically we say that an ordering relation exists among the set's members.
[10] This notion of "because I said so" being a property may sound a bit strange at first, but has great utility and practicality, reflecting how we actually think about sets. It will be made more formal later.
[11] There are subtleties here that are clarified only when we decide in a particular context which of object, property, or set have ontological priority. For example, the definition of set requires both set membership and property to have ontological priority, while the definition of class requires at least one property to have priority (i.e., we can take objects as a primitive concept or define objects as being identified by and distinguishable by their properties).

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class is a class whose objects are not sets. Thus, every set is a class, but every class is not necessarily a set. To put it more informally, for classes, a class selects from some universe of objects those objects that have the class' defining properties. Any notion of class "membership" is thus derived, rather than defined. Thus, excepting under certain conditions (see below), we may say that a class induces a set. The distinction between set and class is subtle, and often confusing for the uninitiated.

We will stick to the term *set* here whenever we are concerned with set theory. A set is like a named conceptual container that exists whether it has any members or not. The members of a set share at least the *asserted* property of set membership. The properties that members of a set are required to share are the defining properties of the set. By contrast, a class is best thought of as a principle that picks out zero or more objects (whether those objects already belong to some set or are just an abstract collection) and that might be put in a collection (possibly but not necessarily a set). Whether the resulting collection is a set or not depends on how set and class are defined in the particular set theory. For us, sets have a priori existence and can be differentiated one from the other by their defining properties, while classes have existence only by virtue of defining properties.

The reader is warned that, to complicate matters further, the terms *set* and *class* do not have universal definitions. What constitutes an example of a class that is not also a set depends on the particular kind of set theory being used, and on details of how the terms *set* and *class* are defined (assuming both are defined). Not all set theories define classes.

In Cantor's ("Cantorian") set theory and most naïve set theories, every collection is a set. Collections are defined from properties that are abstract in the sense that they need not have an associated procedure for determining if some object has the property or not. The collection resulting from *any* specifiable set of properties is called a set. Cantor allows his sets to have not only a *countably infinite cardinality*[12] (e.g., the integers – we can count them in principle, though never finishing), but to have an infinite number of members for each and every countable member (e.g., the real numbers – we cannot even count them). Cantor found this definition necessary so that his set theory could be used as a description of elementary arithmetic. However, this definition permits sets to have an infinite number of sets with an infinite number of members. And these sets of arbitrary, abstract sets leads, once again, to paradoxes. You can then express infinite sets of infinite sets of infinite sets, and so on ad infinitum. Cantor called these strange sets *transfinite sets*, and we say they have transfinite "cardinalities" (not really a cardinality because infinity is not a specific number[13]). Transfinite cardinalities can be ordered in terms of relative "size" nonetheless and so we call them "ordinalities" or simply transfinite ordinals.

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[12] See the discussion of cardinalities below in Section VII.
[13] Cantor (and most set theorists) considered infinity to be a "quantity of fixed magnitude" nonetheless.

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In ZFC set theory (i.e., “Zermelo-Fraenkel axiomatic set theory with the Axiom of Choice), there are no classes – things we would call classes that aren't sets can't even be talked about and that strategy avoids certain paradoxes.

Russell and Whitehead’s set theory (as found in their seminal work *Principia Mathematica*) has both sets and classes and introduced the first distinction between these terms. The definition of set in Cantor’s set theory leads to paradoxes like ***Russell's Paradox***:

> *“If S is the set of all sets that do not contain themselves, does S contain itself?”*

Russell and Whitehead’s set theory prevents Russell’s Paradox from even being expressed by defining such properties as defining a collection other than a set. Thus, not every collection is a set as it was in Cantor’s set theory. The other kind of collection – which has “members” only by virtue of its defining properties – is called a *class*, but is defined not to be a set. In this set theory and in the simple set theory used in our meta-language, sets can only be defined from a universe of previously and independently identified members, each having one or more properties. The defining properties of a specific set must then serve to select a corresponding portion (i.e., a *subset)* – zero or more members – from this universe. If no member of the universe has some a specific property, then including that property among the defining properties of some set is impermissible. By contrast, it need not be the case that every defining property of a *class* be a property of at least one member of the universe. If there are no exemplars in the universe of a member having a defining property of some class, the class is not merely “empty” but ill-defined and there can be no associated procedure for constructing the class.

NBG set theory (i.e., “von Neumann-Bernays-Gödel axiomatic set theory”) also defines both sets and classes. However, this theory defines the terms *set* and *class* somewhat differently from Russell and Whitehead’s set theory with the result that exemplars of classes that are not also sets differ between them.

If a modern set theory (approximately theories defined after Russell and Whitehead) uses both the terms set and class, it will almost always define a set and a class differently. Generally speaking, every finite and discrete (i.e., so they have a finite countable cardinality) collection with defining properties will be both a set and a class. But if the property that defines a class is either (a) impredicative (circular in some sense) or (b) leads to an uncountable cardinality, then the class is usually not a set.

All this will be relevant to database theory only in explaining why we do not want to have impredicative properties. (Notice that I explain herein that properties needing to have an associated decision procedure). For example, we don't want to permit a relation to have an attribute defined on a relation valued domain (“RVD”) <u>and</u> for that domain to be the set of all relations: That's a circular definition! Such definitions lead to paradoxes, which means that certain algorithms the DBMS needs in order to function simply don't exist.

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The Symbols of Set Theory

There are three main types of symbols used in set theory: one for representing sets, one for representing set relationships, and one for representing set members. Symbols representing sets and set members are sometimes called "set variables" in the sense that, throughout a derivation, a given set symbol (set member symbol) can be understood as representing any arbitrary, but specific set (set member, respectively). Assigning a particular set to such a symbol, often called "set assignment", is an interpretive act that occurs outside any set theoretic derivation. Often, set assignment is introduced with a phrase such as "Let" as in "Let A = {0, 1}".

Set membership is characterized by a binary relation[14] (the ***membership relation***) between an member ***o*** and a set ***A***. In this context, the concept of a relation is primitive (and not to be confused with the database term "relation"). If (and only if) this relation between ***o*** and ***A*** is satisfied then the ***o*** is said to be a member of the set ***A***, written $\boldsymbol{o} \in \boldsymbol{A}$.

Specifying A Set: Extensional vs. Intensional

Definitions of a set (or of the set's membership relation) are one of two types, *extensional* or *intensional*. Both types of definition can be given in a formal language such as axiomatic set theory, propositional logic or predicate logic, the expressive power of that language determining the utility of the definition (a point to which we will return in a later article). It should be understood that it is always possible to give an *extensional* definition of a set. Whether one gives an *intensional* definition for a set is a matter of convenience. When it comes to expository power, the definition of choice may be either extensional or intensional and will depend on context.

An ***extensional*** definition of a set explicitly and exhaustively designates the members of the set as a collection. Such a definition is *denotative* (in a philosophical sense) in that every member so designated is an exemplar of the set's members. An extensional definition presumes that the given exemplars are distinguishable each from the other and the given exemplars are exhaustive of the set. In written discourse an extensional definition is often given in set notation as a comma separated list of (symbols representing) members surrounded by curly brackets (braces). For example, the set of integers between one and five inclusive is written as "{1, 2, 3, 4, 5}". Such extensional definitions rely upon the reader's understanding that the symbols used invoke prior knowledge of the referents: The symbols denote either abstract or concrete concepts presumed already known to the reader.

In computing, we often make use of the extensional definition of a set in one of two ways: either

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[14] For those familiar with relational database theory, do not make the mistake of interpreting this as a "set of 2-tuples". Doing so would make the definition circular, rather than treating the concept of "relation" as primitive.

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by maintaining an exhaustive list of the members or by providing a recursive method for generating that list. As an example of the former method, we might store a collection of values or a so-called "lookup table" as the definition of some set[15], and then check to see if some particular value exists in the lookup table to determine whether or not it is a member of that set. Note that this method assumes existence of a computational procedure for testing equivalence. We would iterate through the collection checking for equivalence until either an equivalence is found or the entire collection has been checked.

As an example of the latter method, we might encode the Peano axioms (establishing both zero as a natural number and a successor function on the natural numbers) to generate a list of the integers up to some maximum as needed, and then check to see if the alleged integer is a member of that generated set.

An ***intensional*** definition of a set is a description that permits a set's members to be distinguished, out of a fixed, specified collection (the universe), from non-members in some operational and preferably computable sense. Simply put, an intensional definition offers a test for set membership. Such a definition is *connotative* (in the philosophical sense) in that it provides a principle for determining which objects (of the specified collection) meet the requirements given in the definition. An intensional definition appeals to the notion of properties or attributes of the members of a set, and relationships among them. For example, we might define the set of balls as consisting of those objects that have the properties of three-dimensionality and symmetry in three dimensions (sphericalness).

In addition to simple properties, relationships among properties can be considered properties in their own right. An intensional definition thus specifies a non-empty set of properties that uniquely characterizes the set, each member being distinguishable from other members by at least one non-shared property. Notice that the definition of set requires that we assume the set "exists", even if no object in the specified collection satisfies the intensional definition of the set, the result in that case is still understood to be a set, but is an ***empty set***[16] (i.e., a set with no members)**.**

Just to put a fine edge on these concepts, note that, for the set of positive integers (with zero)[17], we can use a variant of the system of Peano Axioms (PA) as an intensional definition (in addition to the extensional approach given above) by defining a predecessor function. Given a

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[15] The set has no meaningful order, but is it often convenient in both mathematics and programming to impose an order for purposes of explaining or examining the set as a list. Because changing the imposed order does not change the meaning of the set, the ordering is merely an artifact. When manifested in computer storage, it becomes purely physical.
[16] In naïve set theory, there is exactly one empty set. However, in typed (a.k.a., sorted) theories and following Russell, there is one empty set per type. We will have need of typed theories and will assume that empty sets are typed.
[17] I have intentionally avoided the term "non-negative integers" here, which has not place in the Peano construction.

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candidate integer, we can then test to see if repeated application of the predecessor function eventually yields a given integer (such as zero). This procedure is then a computable test of the required property of positive integers – that they are the recursive successors of zero. Under certain reasonable conditions[18], intensional definitions can ultimately be analyzed into some combination of one or more extensional definitions. In practice, of course, expressions entirely in terms of such primitives may be too verbose to be practical and may impede communication, which motivates the use of (controlled) abstraction.

[**Exercise**: For PA, why can't we use the successor function to test a candidate integer instead of the predecessor function?
Answer: PA defines number via induction. As with all cases of induction, only the base object (usually the number 0 here) is defined (Axiom 1) and all other objects are defined relative to the base object using the successor function (Axiom 3). Thus we cannot prove a candidate object is in the set by finding an object that is its successor. We must work backward toward the base, or at least, towards an object already proven to be in the set by applying the predecessor function until that proven object is reached.]

A caution is in order. Sometimes what appears to be an extensional definition will be presented, but in which the exemplars given are not exhaustive. For example, {1, 2, 3, … } is meant to suggest the infinite set of positive integers. Such definitions are, in fact, intensional. They presume that an inductive or recursive definition exists by which those members represented by ellipsis may be generated. In fact such a definition may or may not be known, and even if known, may or may not depend on presumptions incompatible with the formal system under discussion. Readers are well-advised to question such definitions carefully.

III. SETS AND THEIR PROPERTIES

When we speak of properties, we do not mean some undefined, subjective, or imagined concept. A ***property*** is an objectively identifiable concept associated with some object, meaning that there exists an effective and reliable (repeatable) procedure for determining whether or not the property exists (e.g., whether or not an object has the property).

For a given object, either a particular property exists or it does not. In practice, this means that the *property* is *observable* or *measurable*, or is *inferable* from some repeatable observation or measurement procedure. Properties of a set serve to determine whether members of some designated universe are or are not members of that set. For convenience, we will refer to members of the universe that have not as yet been proven (by some means yet to be explained) to be members of some particular set as “candidate” or “potential” members of the set.

[18] In particular, the formal system is required to be *constructive* as defined in the next article.

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Defining Properties

From the beginning, let's distinguish between inherent or intrinsic properties and extrinsic properties. An ***intrinsic property*** is a property that cannot be separated from or added to an object without changing the identity of that object. By contrast, an ***extrinsic property*** is associated with an object, but is not essential to it. The most important and common example of an extrinsic property is a name or other denotation for the object. A denotation is a convention for referring to an object, not an intrinsic property of the object. Denotations for an object can be changed, added, or removed without changing the object's identity.

Sets are collections of objects, called the members of the set, each of which is *required*[19] to have a specific set of one or more properties called the ***defining properties*** of the set. Often the set is itself given ontological significance and a symbol or name given to denote the set in its entirety. However, a set need not have ontological significance – its only import may be as a specification of a collection so that, excepting the fact that objects have been specifically identified or designated as members, the set has no conceptual meaning.

Warning: In the literature, a property is sometimes defined according to its extension. That is, a property is defined according to those objects that have the property. In such treatments, two properties that have the same extension are said to be the same property. Similarly, if two binary relations hold between exactly the same pairs of objects, then such treatments say they are "co-extensive relations" and are the same relation. (As we shall see below, we consider relations to be properties in their own right.) Extensionally defined properties and relations is a useful device in abstract formal systems. However: *We do not embrace the concept of extensionally defined properties, finding that it limits our ability to capture and distinguish meanings, thereby limiting interpretational power disadvantageously.* We believe it is self-evident that we must distinguish between properties that have the same referent (extension) but a different sense. For example, if we limit our UoD to the integers less than nine (9), then the extension of both (a) the property "the smallest even prime" and (b) the property "the cube root of some ***n***" is exactly "2". But these properties have a different *sense* and that difference must not be lost.

One property that deserves special attention is that of *designation*. The sole defining property of a set can be simply that the definer of that set has explicitly designated certain objects (one or more) as all the set's members. Each such member then has the property of having been designated as a member of a specific (e.g., named) set. We will refer to such a property as a ***designating property***. Sets formed in this way may seem somewhat special, but are common nonetheless: The set is implicitly formed by an arbitrary collection of objects. Note that, for such sets, all members must be identified individually and unambiguously in advance else the

[19] If you find the notion of a set property being a requirement, perhaps you are thinking of sets as having some kind of *a priori* existence and their properties as being "discovered". While such a position is fine for pure (abstract) mathematics, it is antithetical to applied mathematics in which sets are defined and often constructed.

definition is impredicative. We will have special use of the *designating property*[20] concept when we address the database topic of specifying a relation in terms of the defining properties of its members.

As an example of a set defined by a *designating property*, consider the contents of a particular paperbag of items or of a particular toolbox. We can easily imagine that the only known property the members of these sets may share is the fact that they have been designated as members of their respective set (the bag or the toolbox), which is then given a name or other sign to denote it. As other examples, imagine someone writes down words that come to mind by free association or a child writes down their favorite things. Were we to find these words without knowledge of their origin, we might reasonably have no awareness of any shared properties other than the fact that they appear together.

Given the foregoing, a set is a collection of objects that are, in a very specific sense, equivalent. To be precise, for every set S there exists a list of effective procedures, one per defining property of the set, that takes as input an object and yields a yes or no answer as output[21]. Note that these effective procedures must be completely independent of each other and of all non-defining properties of the set. Then, each object that is a member of S yields a list of yes/no answers that is identical to the lists produced by every other member. In other words, for every set, the members of that set belong to a certain *equivalence class* defined by the set's defining properties.

Meaning Criteria

Some defining properties are formed as the disjunction of two or more properties. These disjuncts, taken together, are called ***meaning criteria***. Each *meaning criterion* (an individual disjunct) induces a partitioning of the set into two subsets, those that meet the criterion and those that do not. Alternatively, we can say that each *meaning criterion* serves to differentiate a possible subset of a set from other subsets of the set. Notice that some of the possible subsets will be disjoint, while others are not.

Each of the possible subsets of the set is then defined by ("inherits"):
(1) the defining properties of the set conjoined with
(2) at least one meaning criterion (that meaning criterion, or those meaning criteria, thus becoming the defining property, or properties, respectively, specific to the proper subset).

In fact, certain sets may have only properties that are meaning criteria and no agreed upon separate defining properties at all. Members of such sets must have at least one property from

[20] In the article entitled Constructing Relation Predicates, we will have cause to represent a certain primitive designating property (an assertion of membership) in predicate logic, calling it the *assertion predicate*.
[21] If the property is non-atomic, the procedure may initially yield a set of values (observations or measurments), which are further classified into a set corresponding to yes and a set corresponding to no.

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among the list of meaning criteria. The classic example of such a set is the set of all poems. Other examples include the set of all songs, all sculpture, all art, and so on.

We will discuss this topic more when exploring the concept of types and subtypes. <TBD: Add graphic Figure 2.1 below showing relationship among sets/subsets and defining properties vs. meaning criteria. >

Member vs. Collective Properties

Most *defining properties* are properties that belong to the individual members of the set. Keep in mind that all set properties, whether defining properties or meaning criteria, are applied in the context of a UoD. These ***member properties*** can be identified by testing each candidate member of the set (i.e., objects that are potentially members of the set such as those in the UoD) without concern for its relationship to other candidate members of the set. By contrast with member properties, however, sets may also have as defining properties certain ***collective properties***, i.e., properties of the collection, being a relationship that holds among all the members collectively (see the example in the next paragraph).

Every member of the set must be distinguishable from every other member of the set. Both set membership and member distinguishability rely upon the identifiable properties of the set members. This fact leads quite naturally to a common example of a collective property, the count of members in a set, known as the set's *cardinality*. If the set's members are not distinguishable, the members cannot be counted and so the set's cardinality cannot be determined.[22] Neither can the cardinality be determined by examining any single member.

The cardinality of a set is an important example of a property of the collection. Although cardinality is often treated as an implicit consequence of set definition, this need not be the case. In specifying a particular set, we might require that the cardinality of a set satisfy some relationship (including equality, greater than, less than, etc.) to a particular value or to some other quantitative property. Such cardinality relationships are then a defining property of the set. For example, consider some common sets in which cardinality figures prominently in the definition such as a dozen eggs, a gross of pencils, a legion, and so on.

If the only defining properties of a set are one or more collective properties, they often do not suffice to determine a unique collection of objects out of the UoD. Rather, they may only determine an equivalence class[23] (that happens to have sets as its members). Suppose that the

[22] For example and following Cantor, the real numbers are usually considered to be uncountable though they are assigned an ***ordinal*** infinity of $\aleph_1$ greater than the infinity of integers $\aleph_0$.

[23] The reader is reminded that every designated set is a class, but that a class should not be understood as necessarily designating a set. An equivalence class is a collection implicitly defined by the fact that its members share some property or properties. Those members are said to be *equivalent up to* those properties. Unfortunately, the standard term *equivalence class* may or may not be a *class* depending on how class is defined in the particular set theory.

defining properties of some set are that its members be positive integers less than four and that its cardinality be exactly two. Notice that, if these are the only defining properties, that there are then three possible sets ({1,2},{2,3},{1,3}) that have the required properties. In other words, we have defined an equivalence class of sets, but not a unique set.

As another collective property, consider the requirement that every member of a set have a unique value of some specific defining property ("property value uniqueness"). For example, suppose that all the members of our universe (from which set members are drawn) have a color. We might require that no two members of the set we are defining have the same color. If the number of colors is small compared to the number of members of the universe, then there are possibly many sets that would satisfy this requirement. Furthermore, color uniqueness can only be determined by examining and comparing the set's potential members with members already selected. Even then, there will be possibly many sets that satisfy the requirement of color uniqueness. Once again, collective properties – in this case a uniqueness requirement – determine an equivalence class (of sets) and we can only decide whether or not a specific set in that class has the collective property.

Every notion of uniqueness is a collective property, including the alleged intrinsic uniqueness of members of a set. If that uniqueness is to be anything other than an abstraction, we must specify what properties make the members of the set unique and how, rather than merely assume it.

As yet another example of a collective property, we might require that each member of a set is an object that has a dollar value but the set must have some minimum or maximum combined dollar value. Once again, we need to inspect all the members of the set to determine that the set has (or does not have) the collective property. As further examples, note that most statistical and aggregate properties, including average, mean, standard deviation, sum, minimum, maximum, outliers and so on, are collective in nature.

Atomic Properties vs. Qualitative and Quantitative Properties

It is important to remember that, for some properties, only their existence or non-existence can be observed (an object either has the property or it does not). We will refer to such properties as ***atomic properties*** – an object either does or does not have an atomic property. Atomic properties (as contrasted with non-atomic properties) do not have values *per se* but either exist or do not in some context. For example, an object might have the atomic property of greenness. The only "values" that may be associated with an atomic property are dichotomous, such as "true" or "false", interpreted as meaning the atomic property either does or does not exist in the context, respectively.[24]

[24] In passing, we note that atomic properties are closely related to the notion of a bit of information and the theory that the amount of information required to describe anything uniquely is decomposable into a set of yes/no questions.

When a property can be both observed and measured, it is sometimes a ***derived property***, meaning that it can be defined in terms of primitive properties. Derived properties are sometimes said to have, or are assigned, qualitative or quantitative values. This simply means that the definition of the property entails a set of atomic properties, the members of which may or may not have some intrinsic relative order. For example, if the property of "taste" is said to be one of {sweet, sour, bitter, salty}, then by the taste property we mean any of those properties in the set formed by the atomic properties of "sweetness", "sourness", "bitterness" and "saltiness". Similarly, if we say that the property "LessThanThree" means one of {0, 1, 2}, then it is any one of those properties in the set formed by the atomic properties "zeroness", "oneness", and "twoness".

More structure is required to specify non-atomic properties that might, for example, require a member of a set to be "a degree of greenness" (either qualitatively or quantitatively). In particular, a non-atomic property may imply a set of possible values that are either qualitative or quantitative. For example, to say that a non-atomic property can have qualified values such as "very green" vs. "slightly green" or unqualified but related values such as "foolish" vs. "wise" is to say that the property is *qualitative*. By contrasting example, if a non-atomic property is some number in a range of numeric values, such a property is *quantitative*. For example, if the range of "color wavelength" is "495 -570 nanometers"[25] a specific value might be "500 nanometers" or if the range is the set of integers, the specific value might be twelve. As we will see later, in the relational model what I will call (primitive) domain values correspond to atomic properties (i.e., primitive data types), and both derived domains and attributes[26] correspond to derived properties.

Both qualitative and quantitative properties require an associated effective procedure by which a property may be qualified or quantified, thereby deciding whether or not it has a specific value. If qualitative, the possible values are merely an exhaustive list. If quantitative, the possible values are typically recursively enumerable and are put in a one-to-one correspondence with, for example, the natural numbers (up to some maximum). In either case, the associated effective procedure may be computable, may involve a set of real world operations (e.g., physical measurements taken by some mechanism), or may involve judgment (e.g., in the case of subjective ratings).

A Useful Classification of Properties

Defining properties of a set can be classified as first order, second order, third order and so on,

[25] This, as it happens, is the wavelength range for the color is commonly known as "green".
[26] One reviewer pointed out a possible difference between this usage of the term "attribute" and that in *TTM*. Indeed, there is a difference: I use the term attribute as synonymous with property and distinguish the term attribute from the term "attribute value". That said, a domain is a data type.

according to how properties and sets are being used to represent some subject. In other words, what we will call the “order” of a property is not intrinsic to the property, but rather of its use in a representational context. Understanding how to classify a particular property in this manner will be helpful when we learn, later on, how to express the property in a formal language.

If the property in question is *primitive*[27] in the representational context, we say it is a *first order property*. A first order property of a set is a property of each individual member of the set. For example, the simple color property we described above is an example of a first order property. Some properties that pertain to an individual member of a set are not primitive, and signify relationships among first order properties. We will call these *second order properties.* Complex properties can often be expressed as second order properties. Every particular second order property is defined in terms of one or more first order properties, and so has only an indirect relationship to any individual member of the set. For example, color saturation would be an example of a second order property. A property among two or more members of a set is called a *third order property*. When third order properties are among all the members of a set, they are *collective properties* and otherwise, being among less than all the members of a set, they are *partial collective properties*. For example, color uniqueness as described above is an example of a third order property. A fourth order property is a property among sets.

[**Exercise**: Give an example of a fourth order property.]

Simple Sets vs. Complex Sets

Another useful way to categorize sets according to their defining properties is as either simple or complex. We will define a ***simple set*** as one which has only first and second order defining properties. That is, inspection of an object is sufficient to determine whether or not it belongs to the set. By contrast, we define a ***complex set*** as one that includes among its defining properties at least one relationship among the members. For complex sets – which have third or higher order defining properties – inspection of an object (in the UoD) is not sufficient to determine whether or not it is a member of the set: We must also consider relationships among all other members in the set (*collective properties*) and other potential relationships as well.

A trivial example of such a complex set is one that includes among its defining properties a relationship pertaining to set cardinality, such as a specific, minimum or maximum cardinality. Another example of a complex set with third order defining properties is one for which some numeric defining property exists and a relationship involving some aggregation of this property's values is also a defining property. As yet another example, if one of the defining properties is derived property, it is possible that every member of the set is required to have a unique value for that derived property. That uniqueness requirement is, of course, a third order or collective

[27] Recall that the adjective ***primitive*** indicates that it designated as being *a priori* unanalyzable and not defined in terms of anything else. All non-primitive terms must be defined from primitive terms.

property, making the set complex.

Enumerable Sets and Ordered Sets

Unless we state otherwise, the members of a set never have an ordering. However, we will often have reason to consider sets for which the members can be exhaustively enumerated or listed according to some procedure. The procedure might produce the same members multiple times so that the list includes repetition, but ultimately it must list all the members. Sets are called ***enumerable sets*** if such a procedure exists. If the procedure is an algorithm (see *effective procedure* below) we say the set is an ***effective enumerable set***. Not all sets are enumerable, let alone effective enumerable, although most of the sets we shall have cause to consider are enumerable. In fact, ordered sets are an essential part of the apparatus of both mathematics and logic.

While nothing in the definition of a set demands that the members of the set be orderable, nothing in the definition forbids it either. The order or lack thereof of a set's members is completely orthogonal to the concept and definition of set. Contrary to some authors, it is not wrong to speak of an "***ordered set***" (a set whose members have some order) and we shall indeed do so.

The members of a set may be ordered according to an ordering procedure that enumerates the members of the set according to some defining property (or properties). An ***ordering procedure*** is a procedure that (a) enumerates the members of the set, (b) produces a list without repetition, and (c) always produces the same list. Any effective enumerable set may be ordered by supplying each member with an extrinsic property called an index, thereby yielded an ***indexed set***. An index for a set of cardinality ***n*** is a list of ***n*** elements[28] having a one-to-one correspondence with a range of ***n*** integers. The process of ***indexing a set*** consists of assigning one element of the list to each member of the set.

Ordered sets are said to have an ***ordering relation***[29] (often given by the symbol ">" or "<") among its members so that one member may be said to be the *successor* (or *predecessor*) of some other member. If, for every two distinguishable (i.e., ***distinct***) members ***a*** and ***b*** of an ordered set, either ***a*** is the successor of ***b*** (***a*** > ***b***) or ***b*** is the successor of ***a*** (***b*** > ***a***), then we say that the set is a ***totally ordered set*** and that the ordering relation on the members is a ***total ordering***. Every indexed set is a totally ordered set. Alternatively, if for any three distinct members ***a***, ***b***, and ***c***, it is the case that ***a*** is the successor of ***b*** and ***a*** is the successor of ***c*** but ***b*** is not the successor of ***c*** and ***c*** is not the successor of ***b***, then we say that the set is a ***partially***

[28] The terms "member" and "element" are often used interchangeably. I've introduced the term *element* to denote a component of a *list* (i.e., a sequence of symbols) simply for clarity of reference. I reserve term ***element*** to denote a component of any formal structure, with *members* being the reserved term for elements of a set.
[29] Ordering relations are usually defined on arbitrary rather than distinct members, so that the relation includes the possibility of equality of members rather than just being a successor.

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ordered set (or "***poset***") and the ordering on the members is a ***partial ordering***[30].

Ordering relations and ordered sets are very common in mathematics, logic, and computer science. Set inclusion is an ordering relation, as is set membership. Both the real numbers and the integers are totally ordered sets. The set of subsets of a set are partially ordered by inclusion. A list is a totally ordered set. A tree (hierarchy) is a partially ordered set. The definition of any metric requires ordered sets. An important branch of mathematics known as lattice theory – often useful in proving properties of formal logical systems – involves the study of partially ordered sets with a *supremum* (a *least upper bound*) and an *infimum* (a greatest lower bound). Boolean logic corresponds to a particular kind of lattice (a complemented, distributive lattice)[31].

IV. ONTOLOGICAL COMMITMENTS

Traditionally, philosophers and logicians have taken the objects that belong to some theory (e.g., set theory) as primary and have imbued them with an assumption of existence. In short, they have made an ***ontological commitment*** to the primacy of object existence. This has meant that objects are *analyzed* (rather than defined) according to their properties, with a fundamental epistemological problem being to discover and elucidate those properties. Whether that problem was solved or not – and debate regarding it has gone on unabated for millennia – objects have almost always retained the presumption of independent existence. Readers familiar with the philosophical literature will be well aware of the primacy given to objects and of the many ontological difficulties that must then be overcome. For purposes of future reference, let's agree to refer to this as the ***object ontological commitment***.

An even more troublesome assumption often supersedes, and is a specialization of, an object ontological commitment. Applying specifically to set theory, the ***set ontological commitment*** assumes that any conceived (or merely named) set has existence. Set ontological commitment, without the caveat that the members of every set must be defined before that set is defined, leads to paradoxes. In most axiomatic set theories, extra measures are taken to avoid problems. Axiomatic set theories such as ZFC (Zermelo Fraenkel with the Axiom of Choice[32]) assume set ontological commitment, but do not make an object ontological commitment to individuals[33] and, in fact, treat neither individuals nor classes. Other axiomatic set theories like Von

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[30] Formally, a partial ordering (relation) on arbitrary members (not necessarily distinct) is reflexive, antisymmetric, and transitive. A total ordering is a partial ordering in which for every distinct pair ***a*** and ***b***, only one is the successor of the other.

[31] My logo is the diagram for the simplest non-distributive, symmetric, modular lattice. It's asymmetric partner is the lattice corresponding to *quantum logic*. I chose to use the symmetric version for my logo in 1975 for aesthetics.

[32] We note in passing that ZFC permits the construction of infinite sets, without requiring an infinity of individuals (objects that are possibly members but are not sets).

[33] Individuals in axiomatic set theories are sometimes called "ur-elements" (from the German) or "primitive element".

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Neumann–Bernays–Gödel set theory ("NBG") address both sets and proper classes (classes that are not sets), but again make no provision for individuals. Most systems can, however, be modified to allow for individuals but the damage due to an unrestricted set ontological commitment remains: there is no requirement that a set be ultimately constructed from individuals.

In this series of articles, we will ultimately propose and elucidate a different ontological commitment in which properties are given primacy. By contrast, objects – and anything derivable from them – will be treated as having existence only in consequence of their relationship to and dependence upon properties. For purposes of future reference, we'll refer to this as the ***property ontological commitment***. Those readers who have designed relational databases will recognize the importance of properties (captured as attributes and constraints) in relational theory, although traditional explanations of relational theory fail to make this explicit or to capitalize upon it.

In the discussion of properties which follows, consider the differences that result when read in the light of an *object ontological commitment* versus a *property ontological commitment*. In a future article, we will return to these concepts and their very important consequences.

V. DISTINGUISHABILITY

The concept of a set definition hinges on how we understand distinguishability. Every member of a set is, by definition of what we mean by a set, distinct from every other member of the set. To put this another way, there are no "duplicates" or "indistinguishables" in a set. Although seldom made explicit in the literature, Leibniz' Principle of Identity of Indiscernibles, a.k.a. Leibniz' Law[34], is usually assumed to apply as an ontological basis for set theory:

> **Leibniz' Principle (of Identity of Indiscernibles)**
>
> If no property **P** can be discerned by which **x** and **y** differ, then **x** and **y** are identical.
>
> or
>
> For every discernible property **P**, if object **x** has **P** if and only if object **y** has **P**, then **x** is identical to **y**.

To put it another way, if we want to assert that two objects are distinct, we must identify at least

[34] Wilhelm Gottfried Leibniz, Discourse on Metaphysics, Section 9 (Loemker 1969: 308).

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one property that they do not have in common. John McTaggert (1866-1925) phrased a related law called the Dissimilarity of the Diverse in his treatise *The Nature of Existence*: Objects ***x*** and ***y*** are each distinct if and only if there exists at least one property that ***x*** has and ***y*** does not. We shall see when we treat the sets called relations, and especially database design, that Leibniz' Principle is of great importance[35] and value. (Indeed, the Turing Test is an application of it.)

The properties of a set that serve to distinguish its members are not necessarily all of its defining properties. All that is required is that some minimal set of defining properties be *unique* for each member of the set. With respect to properties that establish distinguishability, a set may be *over determined* so that any of several properties may serve to distinguish the members. When distinguishability is over determined, we should suspect that there may be a relationship between those distinguishing properties and at least one is not primitive. Some of these properties are applied to the set member by practice and so are ***extrinsic properties***, while others are inherent and so are ***intrinsic properties***. In the same way, two sets may be mutually distinguishable through more than one of their defining properties. When extrinsic properties are used to distinguish members of a set, they quite possibly are subject to change and we say they are not *stable identifiers*.[36] By contrast, intrinsic properties are stable and so can be used to distinguish without concern that they might change.[37]

For example, the members of a set representing employees might have as properties a unique employee number, unique fingerprints, unique DNA, a unique title and name combination, a unique Social Security Number (SSN) or Taxpayer Identification (TID), and so on, any one of which suffice to make the members of the set distinguishable. In this example, SSN and TID – although having a different *sense* – act like synonyms and so have a one-to-one relationship. Properties like employee number and SSN are extrinsic properties (in this case, names that denote the employee), while properties like DNA and fingerprints are intrinsic properties.

Warning: It is common practice when defining a set in terms of its properties to give such properties as are necessary for the discussion at hand, rather than exhaustively giving all the sets defining properties. At the very least, a sufficient number of properties should be given to distinguish each set in the discussion from other sets in the discussion, and to distinguish each member of a set from every other member of a set. When the discussion is abstract or

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[35] Reasons have been given in the literature for rejecting Leibniz' Principle, but are not relevant to our subject. We do note that, although Leibniz' Principle requires second order predicate logic to formalize ($\forall P\ (P(x) \sim P(y))$), for our purposes it is ontological and therefore need not be formalized. It is introduced into the first order axiom system simply by including equality as a primitive (i.e., unanalyzed) predicate having the properties of symmetry, reflexivity, transitivity, and replacement.

[36] We shall see that at least one stable identifier is necessary for the concept of *key* in relational database theory.

[37] We are not referring to change here in the sense that a property of a dynamic system changes in some specifiable manner. Such properties can be given so they are time invariant – changes with time are part of their description. Note also that a property that changes over time in a random way cannot be a defining property, although the fact that every member of a set has a property that changes in some random way can be a defining property.

mathematical, mathematicians and logicians often fail to specify an inherent property that can distinguish individuals, giving instead either a unique symbol for each individual or a symbol with a unique number (often written as a subscript) called an "index". Neither the cardinal value nor ordinal value of an index has significance – it is merely a way of keeping track of presumably distinguishable individuals.

Various interpretations of Leibniz' Principle reappear in logic and axiomatic set theories in the form of axioms. For example, and for readers interested in axiomatic set theory, related axioms from set theories or their analogous logics include the:

- *Axiom of Choice* –there exists a choice function for every finite set[38]
- *Axiom of Comprehension* (naïve set theory) – every property determines a set[39]
- *Axiom of Restricted Comprehension* (e.g., ZF and ZFC) – for every property there exist disjoin sets that differ exactly by their members having or not having that property[40] (a.k.a. Axiom of Separation or Axiom of Specification)
- *Axiom of Class Comprehension* (e.g., NBG) – *approximately*, every atomic property (or combination of atomic properties) determines a class and vice-versa[41]
- *Axiom of Extensionality* – two non-empty sets have the same members if and only if they are the same[42].

In the context of set membership, the distinguishing properties of objects are often de-emphasized, while the common properties are emphasized because they define the set. By ignoring those properties that serve to differentiate a set's members, we are defining an ***equivalence class***. That is, with respect to the common properties, the set's members are precisely those that are equivalent when just those properties are compared. One must keep in mind, however, that members of a set have both properties in common and unique, differentiating properties. Naïve set theory (and even elementary set theory) is often taught without pointing out that, while the members of a set are not identical, they do form an equivalence class. Because the common properties of set members can leave a more indelible

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[38] Think of a choice function as a kind of partial inverse membership function: Given the set, its members can be chosen individually. Contrast this with a membership function: Given individual objects, those that are members of a set can be distinguished from those that are not.
[39] The (unrestricted) Axiom of Comprehension leads directly to Russell's Paradox. Herein, we assume that this is not what is intended by Leibniz' Principle, nor implied by it.
[40] The Axiom of Restricted Comprehension precludes Russell's Paradox in ZFC, ZF, and related systems.
[41] Stating the Axiom of Class Comprehension precisely requires concepts in predicate logic we do not yet have.
[42] The caveat that the sets be non-empty is required in case the members are individuals – empty sets are equal only if and only if the specifications of those empty sets are identical.

impression than the requirement of their distinguishability, we must be vigilant in characterizing set membership and not ignore distinguishing properties.

In the context of set differentiation, Leibniz' Principle says that every set is distinguishable from every other set by at least one property. In the simplest interpretation, this means that no two sets have identically the same members. The version of set theory most often used in foundations of mathematics is the Zermelo-Fraenkel axiomatization of set theory which includes the Axiom of Choice ("ZFC"). In ZFC, Leibniz' Principle is captured through the combination of the *Axiom of Extensionality* (the identity of a set is fully determined by its members) and the *Axiom of Restricted Comprehension* (any definable "subclass" – collection defined entirely by properties – of a set is also a set)[43].

It is sometimes argued that some classes of objects (albeit abstract) are inherently and mutually indistinguishable. Even if we entertain this idea, then such objects are not members of any set. Hence, at least for the purposes of set theory and related formal systems, indistinguishability of distinct objects is to be rejected.[44] In computing, we often fail to identify all the attributes or properties of objects with the result that they are given distinct representation in storage but no distinct representation otherwise, and then erroneously assert that they are indistinguishable yet distinct.

For example, in a computer program we typically represent objects using named variables in a computer language. Declaring a named variable in a typical computer language typically reserves a name in a symbol table which, at some point in time, will be bound to a storage address and a certain amount of storage will be allocated to it. Eventually, a value will be assigned to the name, and that value stored in the allocated storage.

Depending on the language, the variable may have a type, meaning it has certain properties. However, there is no requirement that all the properties of the object that is represented by the variable be explicitly given. For a simple variable, such a value at best represents a single property of some object which need not be the property that distinguishes it from similar objects. In consequence, it is possible to use two distinctly named variables to represent two distinct objects, and then erroneously assert or conclude that the represented objects are indistinguishable just because the variables representing them are value equivalent. A similar example of such erroneous thinking (and practice) can be constructed if the variable is instead a data structure having $\boldsymbol{n}$ parts (e.g., an $\boldsymbol{n}$-tuple).

Of course, unless we have some means of distinguishing objects, we cannot know that there exists a plurality of objects – we cannot count them. Similarly in physics, certain elementary

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[43] . The Axiom of Restricted Comprehension replaces the (unrestricted) Axiom of Comprehension of naïve set theory.

[44] To be clear, there *are* abstract formal systems in which the concept of indistinguishable is legitimate.

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particles are said to be indistinguishable. However, this use of the term usually means "up to location or orientation." So called indistinguishable elementary particles[45] are countable only because they may be distinguished by their spatio-temporal location or orientation, which are, of course, properties in their own right.

VI. MEMBERSHIP FUNCTIONS AND DECISION PROCEDURES

The foregoing permits us to formulate a "computable test" or "criterion for membership" in a particular set, called a ***membership function***, which behaves as a definition of the set. Typically a membership function is intensional but is occasionally extensional. It is a Boolean function of the object, returning either TRUE (or other indication of "is a member of set **A**") or FALSE (or other indication of "is not a member of set **A**"). A membership function is often called by various other names including *indicator function*, *membership criterion*[46], and *characteristic function*. As we will see when we discuss relational databases later in this series, every relation must have a membership function (i.e., that relation's defining predicate – its *relation predicate*).[47]

Formal problems having a yes-no answer are called ***decision problems***. The determination of whether an object is or is not a member of some set is an example of a decision problem. Often decision problems pertain to whether or not some set of one or more properties exist. A computational method or algorithm that solves a decision problem is called a ***decision procedure***[48]. Thus a membership function provides a computational method or algorithm for solving a particular decision problem, in this case whether an object is or is not a member of the set in question. When a decision procedure exists for solving the decision problem with respect to some alleged property, the decision problem is said to be ***decidable***. When the alleged property is set membership, then the decision problem has associated with it a set of simultaneous inputs,[49] one for each defining property of a set. Each object is tested by making a set of one or more value-to-parameter assignments in the decision procedure. The decision procedure then classifies each set of assignments into one of two sets. Solving a decision problem for set membership is therefore equivalent to the challenge of finding a decision

[45] Electrons are sometimes said to be indistinguishable. Nonetheless, in addition to being distinguishable by spatio-temporal location, each contributes a unit of charge and the total charge may be measured to determine the number of electrons in some volume of space.
[46] To be clear, a membership criterion, albeit singular, need not be monolithic. It will often have components, but be used as a unit.
[47] C. J. Date, to whom I proposed this concept, and with whom I formalized it in early 1994, now uses the term *relation constraint*. He also uses the term *relvar predicate* (a term with which I am extremely uncomfortable) to refer to a psychological concept - the user's mentally held intended meaning of the relation (a concept I find both irrelevant and misleading to practitioners).
[48] The term procedure as used here need not imply a computer procedure.
[49] If the decision procedure is in fact a computer language function, we would call the inputs parameters, so that a set of assignments to those parameters become arguments.

procedure that serves as the characteristic or membership function of the set.

Aside: *In this discussion, the fact that the membership of a set is often characterized informally, or with a bit of hand-waving, is being ignored (and is irrelevant). The practice may be fine for informal discussion, conveying set concepts through informal but familiar examples, and for certain philosophical pursuits. It has no place in a formal treatment, especially one in which we ultimately want the formal system to be computable (i.e., to be reducible to computational methods).*

A procedure (such as a decision procedure) is said to be *effective* (an ***effective procedure***[50]) for an intended class of problems if and only if, when applied to that class of problems it:

- consists of a finite number of exact, finite instructions[51],
- terminates after a finite number of steps,
- on termination, produces exactly one of possibly several permissible answers (e.g., either TRUE or FALSE and no other answer),
- can be implemented both manually (at least in principle given enough time, paper, ink, etc.) and mechanically (no appeal to an oracle, intuition, or randomness), and,
- requires no ingenuity (i.e., inventiveness) to follow the instructions (only rigor).

The notion of effective procedure can be formalized in various ways to make it applicable to mechanical computation. The class of procedures that have the property of being ***effectively computable*** (a.k.a. *recursively computable*) is formally known as the class of *general recursive functions*. The terminology of recursion theory, especially as it applies to computability, will be discussed in more detail in a later article.

In general, we cannot be certain that a particular procedure will halt (terminate) for any given input in a class of inputs or run forever. Whether or not some particular procedure (or program) will halt is called the *Halting Problem*. We want decision procedures to halt. If a decision procedure is applied to a problem outside its intended class of problems, we might require not only that it halt, but that it not return any result (other than possibly that an indication that it has halted intentionally).

Closely related to the concepts of decision procedure and equivalence class is the notion of ***classification functions***[52]. A classification function[53] may be thought of as a generalized

[50] Caution: An effective procedure, albeit an algorithm, is not necessarily a *mathematical* algorithm.
[51] The term "instruction" should not be interpreted here as implying either a computer instruction or a step in a mathematical algorithm.
[52] One reviewer has, astutely, asked why classification functions are not called classification procedures: this is merely a matter of historical accident.

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decision procedure that combines the decision procedures for a collection of sets into a single procedure. It identifies, for a set of specific input values pertaining to some object (and representing the properties or attributes of that object) and a collection of predefined sets, the sets to which that object belongs. Programmers often use classification functions for a variety of purposes such as bucketing (e.g., in statistical analysis), partitioning, hashing, object recognition, and so on. In relational database theory, it is desirable that the DBMS be capable of classifying tuples into relations. In commercial products, this classification depends on users identifying the relevant relation by name. We will discuss other alternatives in a future article.

A ***classifier*** as encountered in field of AI (Artificial Intelligence) is a kind of generalized classification function in that it may include heuristics, the algorithm may self-modify over time according to some measure of effectiveness (i.e., learn), its classifications may be probabilistic (i.e., a measure of confidence may be associated with the outputs) rather than deterministic, etc.[54] Unlike a classification function, a classifier with ***n*** parameters attempts to identify a collection of sets (typically limited to some relatively small number) that will permit classification of a collection of inputs (***n*** simultaneous value-to-parameter assignments) so that the collection of sets is mutually disjoint or, at least, the intersections are minimized in some way.

VII. SOME SET PROPERTIES, RELATIONSHIPS, AND OPERATORS

For a particular formal use (i.e., derivation, proof, etc.) of set theory, the members of all sets are drawn from a single ***Universe of Discourse (UoD)***. We can define the UoD in some specific context as the set union of all the sets that will ever be defined for that context. This can serve as a practical definition for many purposes where a UoD might not be identifiable otherwise. If the UoD is not defined in a set theoretic discussion or application, then the discussion or application is at least ambiguous, if not altogether invalid. The old saying “you can't compare apples to oranges” can be used to illustrate the import of a defined UoD. If the UoD is apples, then no set of oranges can be defined. Likewise, if the UoD is oranges, then no set of apples can be defined. In neither case can sets of apples be compared to sets of oranges. On the other hand, if the UoD is defined as fruit, then such comparisons can be made in terms of their common fruit properties.

When we refer to the size of a set **A** herein, we mean the number of members of the set, called its ***cardinality*** and denoted “**|A|**”. If a set is smaller than the set of natural numbers, we say it has ***finite cardinality***. If a set is the same size as the set of natural numbers, we say it has ***countably***

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[53] We use the term function herein as a mapping between two sets. While it is true that mathematical relation theory can be used to represent the concept of a function, we find that representation unhelpful in this context and, in fact, that it involves circular definitions in which the concept of set is not primitive.
[54] We use decision procedures constantly in daily life (e.g., with every “identification”) and in the most sophisticated pursuits. Even identification of fundamental particles from collider data (particle physics) relies on a decision procedure.

infinite cardinality. If the set is of larger size than the set of natural numbers, we say it has ***uncountable cardinality***. With the exception of finite cardinality, cardinalities are given special symbols representing relative *orders* of infinity: they are *not* numbers in the usual sense. As we will see, the sets discussed in the context of databases will almost always be of finite cardinality. Be forewarned that giving a formal definition of finite cardinality (let alone non-finite cardinality) requires a powerfully expressive language – it cannot be done in the language of elementary set theory.[55]

In simple set theory, the members of sets can be individuals rather than sets and ultimately all sets are constructed from individuals. This understanding means that we can define set relationships and operators in terms of their members. A binary relation called ***set inclusion*** (a.k.a. ***set*** *containment*[56]) exists between any two (not necessarily distinct) sets if the members of one set are entirely shared with the other set. By definition, every set includes itself. If every member of a set **B** is also a member of a set **A**, **B** is said to be a ***subset*** of **A** written $\mathbf{A} \subseteq \mathbf{B}$. Conversely, **A** is said to be a ***superset*** of **B**. If there is then at least one member of **A** that is not a member of **B**, **B** is said to a ***proper subset*** of **A**. Conversely, **A** is said to be a ***proper superset*** of **B**.

The meaningfulness of a set inclusion relationship depends on there being an implied hierarchy among the intensional definitions of the involved sets. Note that if **B** is a ***proper subset*** of **A**, the intensional definitions of **A** and **B** are related to each other in a specific manner. In particular, each member of **B** must have a defining property that may be used to distinguish it from other members of **A**, and it must have all the defining properties of members of **A**. Thus, the intensional definition of a subset **B** is the intensional definition of **A**, augmented with the requirement of at least one additional defining property.

[Exercise: If a set **A** has **n** defining properties, how many subsets of **A** can be given an intensional definition? How many ways can these subsets be organized into hierarchies? What are the implications for programming methodologies or database management systems that attempt to model an application by set inclusion? Can you identify such a methodology that takes such an approach?]

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[55] The Peano axioms do not belong to elementary set theory. While we can exhibit a sequence of sets that can be put in one-to-one correspondence with the natural numbers, we cannot use the axioms of elementary set theory to define the cardinality of an arbitrary set.

[56] Notice that set containment does not mean the same thing as member containment, the relationship between a set and its members. However, a set may contain members which are themselves sets and those sets may contain members which are sets, and so on. This construct is what is known as a containment hierarchy. In other words, the use of the term containment depends on what you consider to be the members of what.

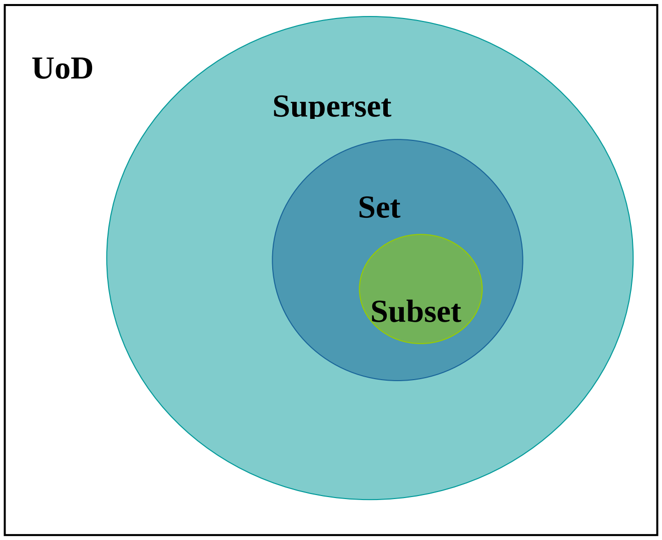

Figure 2.2: Set Inclusion Relationships

Set theory defines a number of dyadic operators on sets including union, intersection, absolute complement[57], relative complement (or difference), and symmetric difference. The ***union*** of sets **A** and **B**, written **A** ∪ **B**, is defined as the set of all objects that are members of **A**, **B**, or both. Suppose set **C** is defined as **A** ∪ **B**. Defining union in terms of the set membership relation, we have, for every member ***c*** ∈ **C**, ***c*** ∈ **A** or ***c*** ∈ **B** or both. (Note that we could also have defined the union of sets **A** and **B** as a set having as its members the sets A and B. This construction is relevant for our simple set theory but will be considered in a later article.) A set **C** is the ***intersection*** of sets **A** and **B**, written **A** ∩ **B**, if **C** is the set of all objects that are members of both **A** and **B** (in terms of the set membership relation, ***c*** ∈ **A** and ***c*** ∈ **B**).

The ***absolute complement*** of **A** includes all members of the **UoD** that are not members of **A**, written ¬ **A** whenever a fixed **UoD** can be taken as understood. Unfortunately, the notation does not reflect the dependence on **UoD**. We write "∉" to mean "is not a member of", so if set **C** is ¬ **A**, then if ***c*** ∈ **C**, ***c*** ∉ **A**. The ***relative complement*** of a set **B** with respect to set **A**, denoted $\mathbf{B}^{\mathbf{A}}$ (or **A\B**) is the set of all members of **A** that are not members of **B**. In terms of the set membership relation, if set C is $\mathbf{B}^{\mathbf{A}}$, then if ***c*** ∈ **C**, then ***c*** ∈ **A** and ***c*** ∉ **B**. Notice that relative complement, like absolute complement, is often treated as a monadic operator, but with **A** taken as a kind of

[57] We note in passing that absolute complement is usually presented as a monadic operator. In texts which assume an abstract set theory having only a "universal" fixed UoD, this practice may be acceptable. However, it does not generalize either to typed theories or to applied set theory in general and so deliberately avoid "monadic complementation" here.

"local" **UoD**. Again, we disparage this practice, preferring to treat both as dyadic operators. The standard *dyadic* operator corresponding to relative complement is called the ***set difference*** of **A** and **B**, written **A – B**. In terms of the set membership relation, it has the same definition as relative complement.

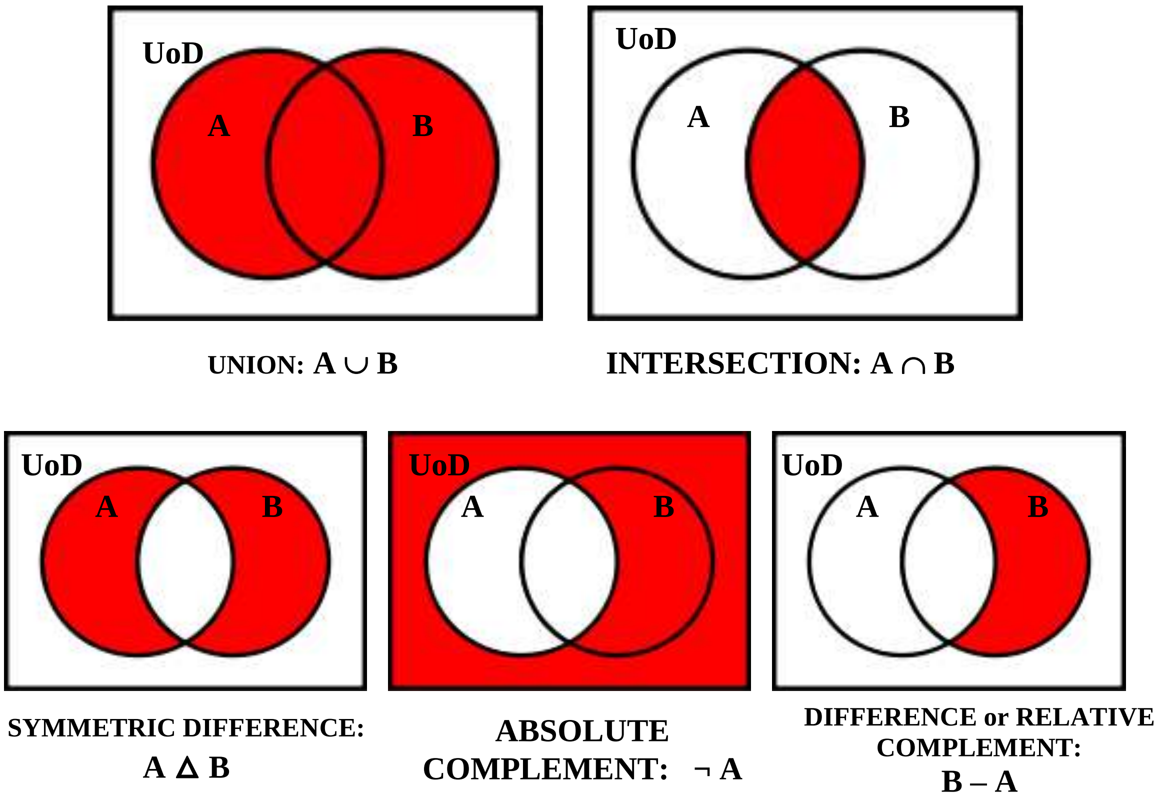

Figure 2.3: Venn Diagrams of Set Operators

A Caution: Writers often refer to the complement of a set **A** without specifying the set it is with respect to, by which they usually mean **UoD – A**. It is important to note that set complement cannot be given interpretation without knowledge of the set to which it is relative, such as the Universe of Discourse. Very often, informal use of set theoretic terminology is made when speaking of multiple domain of discourse with the inclusive **UoD** not clearly defined. In such discussion, complement almost always means a relative complement, with context implying (perhaps ambiguously) to which domain of discourse the complement is relative. Failing to take this into account can lead to anomalies if "laws" or tautologies of set theory are not interpreted with the same **UoD** throughout. For example, when speaking of the complement to a particular set of apples, we usually do not mean the difference between that set and the UoD (which might

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be fruit or even all objects in the empirical world), but rather all apples that are not in the particular set. Unfortunately, when we consider the use of set theory in a relational database context, we will see that the complement is rarely well-defined by writers and methods to rectify this problem will be discussed.

———————

The ***symmetric difference*** of **A** and **B** is defined as the union of the set difference of **A** and **B** and the set difference of **B** and **A,** written symbolically as **A** △ **B** which is equivalent to (**A** – **B**) ∪ (**B** – **A**). These operators are illustrated as Venn Diagrams in Figure 2.3 below[58].

[Exercise: Write out symmetric difference in terms of the set membership relation. Hint: Use two intermediate sets and then form their union.]

In most (but not all!) versions of set theory, every set **S** has associated with it a set called its *power set*. The ***power set*** $\mathscr{P}$(**S**) of a set **S** is defined as the set having as its members every subset of S. For some specific set theories, the power set includes only the proper subsets of **S**. If we should have need of a power set in this series, we will state whether all subsets or only proper subsets are intended in the particular context.

Set Assignment and Set "Variables"

As with most formal systems of mathematics or logic, any derivation or deduction in set theory starts with any set assignments that will be necessary. Set assignment is an initial substitution or assertion of symbolic representation (e.g., "Let **A** = {1, 2, 3, … }" – the set of positive integers)[59] and not to be confused with the kind of assignment that occurs in programming languages (which involves recursion). Technically, such assignments occur outside of any derivation. In particular, a set theoretic assignment is a *rule of correspondence,* (a term we define later) which serves as a definition.[60] Within a derivation, no re-assignment or redefinition can occur (e.g., "**A** ≝ **A** ∪ {…, -3, -2, -1 }" is not permissible, the latter set being the set of negative integers) as this would establish a contradiction. That is, once a symbol has been given an assigned meaning, that meaning is fixed and cannot be changed. The closest concept to "assignment" within a set-theoretic derivation is an assertion of equivalence of two set theoretic expressions.

Within any (first order) formal set theory, its *object language* contains no construct whose meta-linguistic or canonical semantics corresponds to altering an assignment akin to that which occurs

[58] Source of diagrams: Wikipedia - http://en.wikipedia.org/wiki/Venn_diagram. (2014)
[59] As we will see, such assignments are semantic: They provide an interpretation of the abstract formal system.
[60] We will use the symbol "≝" to indicate "is defined as" although the more common symbols used for this purpose in math and logic are "≔" and "≐". Computer science usurps the symbol "≔" to mean "is assigned the value", often meaning destructive assignment (a.k.a., "is redefined as") which is not permitted in set theory or logic.

in programming. To create such semantics requires a non-set theoretic language and one with radically different *rules of inference* than those of the familiar set theories we have been discussing (all of which are called *first order theories*). To put it another way, if value assignment like that found in computer programming languages is added to set theory, the resulting formal system is no longer a set theory in the usual sense.

VIII. CONCLUSIONS

Ordinary English and naïve set theory, augmented with some concepts of elementary set theory as discussed above, will serve us as a meta-language for explaining and discussing formal systems in general, and formal logical systems in particular. Although the presentation of set theory as a formal system in its own right is beyond the scope of this series, it should be clear that such a goal would be feasible and, indeed, many texts exist that achieve that goal. As noted earlier, ZFC and its variants are the axiomatizations of set theory most often encountered in the studying the foundation of logic and mathematics. However, NBG (Von Neumann–Bernays–Gödel) axiomatic set theory with modifications for support of individuals and restricted to finite sets and classes is perhaps better suited to the study of logic as applied to database theory. Readers interested in axiomatic set theory are encouraged to explore NBG and other systems, especially those that avoid set theoretic paradoxes.

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